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Fixed point theorem of Leggett–Williams type and its application

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Abstract

We establish a norm-type cone expansion and compression fixed point theorem for completely continuous operators. Our theorem is then applied to prove the existence of positive solution of second order three-point boundary value problem.

Keywords: Positive solution; Fixed point theorem; Cone

1. Introduction

One of the most frequently used tools for proving the existence of positive solutions to the integral equations and boundary value problems is Krasnoselskii's theorem on cone expansion and compression and its norm-type version due to Guo (see [6]). For applications of Krasnoselskii's theorem to various problems see for example [2,4,5,10,12,13] and the references therein. In [9] Leggett and Williams obtained a generalization of Krasnoselskii's original result and applied their fixed point theorem to the nonlinear equation modelling certain infectious diseases. In the present paper we state and prove a fixed point theorem of

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Leggett–Williams type and give an application of our result to the following second order three-point boundary value problem:

$$\begin{cases} x''(t) + f(t, x(t)) = 0, \\ x(0) = 0, \ \alpha x(\eta) = x(1), \end{cases} \tag{1}$$

where $t \in [0, 1]$, f is a continuous function, $\eta \in (0, 1)$, $\alpha \geqslant 0$, and $1 - \alpha\eta > 0$.

2. Fixed point theorem

We begin this section with reviewing some facts on cone theory in Banach spaces. A nonempty subset P of a Banach space E is called a cone if P is convex, closed and

(i) $\mu x \in P$ for all $x \in P$ and $\mu \geqslant 0$,
(ii) $x, -x \in P$ implies $x = \theta$.

It is known that every cone P induces a partial order in E by

$$x \preccurlyeq y \quad \text{if and only if} \quad y - x \in P.$$

We will write $x \not\preccurlyeq y$ if $y - x \notin P$ and $x \prec y$ if $y - x \in P \setminus \{\theta\}$. We say that P is a normal cone in E if there exists a positive number γ such that

$$\theta \preccurlyeq x \preccurlyeq y \quad \text{implies} \quad \|x\| \leqslant \gamma \|y\|. \tag{2}$$

The smallest γ satisfying (2) is called the normal constant of P. Clearly, for any normal cone we have $\gamma \geqslant 1$. For $u_0 \in P \setminus \{\theta\}$ let $P(u_0) = \{x \in P\colon \lambda u_0 \preccurlyeq x \text{ for some } \lambda > 0\}$. Moreover, for $R > 0$ put $\bar{P}_R = \{x \in P\colon \|x\| \leqslant R\}$.

In 1980, Leggett and Williams proved the following generalization of Krasnoselskii's theorem (see [9, Theorem 2]).

Proposition 1. *Let $F : \bar{P}_R \to P$ be a completely continuous operator with $F(\theta) = \theta$. Suppose there exist r, $0 < r < R$, and $u_0 \in P \setminus \{\theta\}$ such that*

$$Fx \not\preccurlyeq x \quad \text{if } x \in P(u_0) \text{ and } \|x\| = r.$$

Suppose further that for each $\varepsilon > 0$

$$(1 + \varepsilon)x \not\preccurlyeq Fx \quad \text{if } x \in P \text{ and } \|x\| = R.$$

Then F has a fixed point in P with $r \leqslant \|x\| \leqslant R$.

In [1] and [6] one can find some refinements of Proposition 1. Here we recall the result from [1].

Proposition 2 [1, Theorem 1.3]. *Let P be a cone in a Banach space E and $r_1, r_2 > 0$, $r_1 \neq r_2$, with $R = \max\{r_1, r_2\}$, $r = \min\{r_1, r_2\}$. Let $F : \bar{P}_R \to P$ be a completely continuous operator such that*:

(iii) $x \not\preccurlyeq Fx$ *for* $x \in P$ *and* $\|x\| = r_1$,

(iv) *there exists* $u_0 \in P \setminus \{\theta\}$ *such that* $Fx \not\preccurlyeq x$ *for* $x \in P(u_0)$ *and* $\|x\| = r_2$.

Then F *has at least one fixed point* $x^* \in P$ *with* $r \leqslant \|x^*\| \leqslant R$.

Notice that the assumption (iii) can be replaced by

(v) $\|Fx\| \leqslant \|x\|$ for $x \in P$ and $\|x\| = r_1$

(see, for example, [6]).

In this paper we are interested in replacing (iv) by the condition of the norm type, namely

$$\|x\| \leqslant \|Fx\| \quad \text{for } x \in P(u_0) \text{ and } \|x\| = r_2.$$

Remark. In [3], Anderson and Avery obtained a generalization of Krasnoselskii's fixed point theorem of the norm type. They replaced the norm type assumptions by conditions formulated in the terms of two functionals.

In our considerations we will use the following theorem on fixed point index for completely continuous operators in cones.

Proposition 3 [6, Lemma 2.3.2]. *Let* Ω *be a bounded open set in a Banach space* E *and* $P \subset E$ *be a cone. Let* $A : P \cap \bar{\Omega} \to P$ *and* $B : P \cap \partial\Omega \to P$ *be completely continuous operators. Suppose that*

(v) $\inf_{x \in P \cap \partial\Omega} \|Bx\| > 0$,
(vi) $x - Ax \neq tBx$ *for* $x \in P \cap \partial\Omega$ *and* $t \geqslant 0$.

Then $i_P(A, P \cap \Omega) = 0$, *where* $i_P(A, P \cap \Omega)$ *denotes the fixed point index of* A *on* $P \cap \Omega$ *with respect to* P.

For the details of the fixed point index theory in cones see, for example, [6]. Now we can state and prove our main result.

Theorem 1. *Let* P *be a normal cone in* E *and* γ *be the normal constant of* P. *Assume that* Ω_1 *and* Ω_2 *are bounded open sets in* E *such that* $\theta \in \Omega_1$ *and* $\bar{\Omega}_1 \subset \Omega_2$. *Let* $F : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$ *be a completely continuous operator and* $u_0 \in P \setminus \{\theta\}$. *If either*

(vii) $\gamma\|x\| \leqslant \|Fx\|$ *for* $x \in P(u_0) \cap \partial\Omega_1$ *and* $\|Fx\| \leqslant \|x\|$ *for* $x \in P \cap \partial\Omega_2$, *or*
(viii) $\|Fx\| \leqslant \|x\|$ *for* $x \in P \cap \partial\Omega_1$ *and* $\gamma\|x\| \leqslant \|Fx\|$ *for* $x \in P(u_0) \cap \partial\Omega_2$

is satisfied, then F *has a fixed point in the set* $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Proof. We can assume that F has no fixed points on $P \cap \partial\Omega_1$ and $P \cap \partial\Omega_2$. Suppose that (vii) is satisfied. It is known that if $\|Fx\| \leqslant \|x\|$ for $x \in P \cap \partial\Omega_2$, then

$$i_P(F, P \cap \Omega_2) = 1 \tag{3}$$

(see [6]). Next, we will prove that for $t \geqslant 0$ and $x \in P \cap \partial\Omega_1$,

$$x - Fx \neq t(u_0 + Fx). \tag{4}$$

Suppose, for the contrary, that there exist $x_0 \in P \cap \partial\Omega_1$ and $t_0 > 0$ such that

$$x_0 - Fx_0 = t_0(u_0 + Fx_0). \tag{5}$$

Observe that

$$t_0 u_0 \preccurlyeq t_0(u_0 + Fx_0) + Fx_0.$$

This, together with (5), gives $x_0 \in P(u_0)$. Moreover, from (5) we get

$$\theta \preccurlyeq x_0 - Fx_0 - t_0 Fx_0.$$

Hence

$$\theta \preccurlyeq (1 + t_0)Fx_0 \preccurlyeq x_0.$$

Since P is a normal cone, we obtain

$$\big\|(1 + t_0)Fx_0\big\| \leqslant \gamma\|x_0\|,$$

and, in consequence,

$$\|Fx_0\| < \gamma\|x_0\|,$$

which contradicts (vii). Furthermore, for $x \in P \cap \partial\Omega_1$,

$$\theta \prec u_0 \preccurlyeq u_0 + Fx.$$

Hence

$$0 < \|u_0\| \leqslant \gamma\|u_0 + Fx\|,$$

because the cone P is normal. Therefore

$$\inf_{x \in P \cap \partial\Omega_1} \|u_0 + Fx\| > 0. \tag{6}$$

Taking $A = F$ for $x \in P \cap \Omega_1$ and $Bx = u_0 + Fx$ for $x \in P \cap \partial\Omega_1$ in Proposition 3, we get in view of (4) and (6),

$$i_P(F, P \cap \Omega_1) = 0. \tag{7}$$

By (3), (7), and the additivity of the fixed point index we obtain

$$i_P\big(F, P \cap (\Omega_2 \setminus \bar{\Omega}_1)\big) = 1,$$

which implies that F has a fixed point in the set $P \cap (\Omega_2 \setminus \bar{\Omega}_1)$. If (viii) holds the proof is similar. $\square$

3. Existence of positive solution of (1)

In this section we will study the existence of positive solution of (1). Similar problems have been considered for example in [7,8,11]. Particularly, in [8] Infante and Webb proved the existence of multiple nonzero (not necessarily positive) solutions, when either $0 \leqslant \alpha < 1-\eta$ or $\alpha < 0$. It is easy to verify that the kernel of the integral equation related to (1) is given by

$$k(t,s) = \begin{cases} \begin{cases} \left(1 - \frac{1-\alpha}{1-\alpha\eta}s\right)t, & 0 \leqslant t \leqslant s \leqslant 1, \\ \left(1 - \frac{1-\alpha}{1-\alpha\eta}t\right)s, & 0 \leqslant s \leqslant t \leqslant 1, \end{cases} & s \leqslant \eta, \\ \begin{cases} \frac{1-s}{1-\alpha\eta}t, & 0 \leqslant t \leqslant s \leqslant 1, \\ s - \frac{s-\alpha\eta}{1-\alpha\eta}t, & 0 \leqslant s \leqslant t \leqslant 1, \end{cases} & \eta \leqslant s. \end{cases} \tag{8}$$

Observe that for $\eta \in (0,1)$, $\alpha \geqslant 0$ and $1-\alpha\eta > 0$ we have

$$\bigwedge_{t,s\in[0,1]} Mk(s,s) \geqslant k(t,s) \geqslant 0 \tag{9}$$

and

$$\bigwedge_{t,s\in[0,1]} k(t,s) \geqslant m(t)k(s,s), \tag{10}$$

where

$$M = \max\left\{1, \frac{\alpha(1-\eta)}{1-\alpha\eta}\right\} \quad \text{and} \quad m(t) = \min\{t, 1-t\}$$

for $t \in [0,1]$. Suppose that:

(1°) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function and there exists $r_1 > 0$ such that

$$f(t,x) \leqslant \left[M \int_0^1 k(s,s)\,ds\right]^{-1} r_1$$

for $t \in [0,1]$ and $x \in [0, r_1]$;

(2°) there exist $t_0 \in (0,1]$, $a > 0$, $r_2 > 0$, $r_2 \neq r_1$, and the continuous functions $g : [0,1] \to [0,\infty)$, $h : (0, r_2] \to [0,\infty)$ such that

$$f(t,x) \geqslant g(t)h(x)$$

for $t \in [0,1]$ and $x \in (0, r_2]$, $h(x)/x^a$ is nonincreasing on $(0, r_2]$ and

$$\frac{h(r_2)}{M^a} \int_0^1 k(t_0,s)g(s)m^a(s)\,ds \geqslant r_2.$$

Theorem 2. *If the assumptions* (1°) *and* (2°) *are satisfied, then the problem* (1) *has a positive solution on* [0, 1].

Proof. Consider the Banach space $C[0,1]$ and the set

$$P=\left\{x\in C[0,1]\colon\ x(t)\geqslant\frac{m(t)}{M}\|x\|,\ t\in[0,1]\right\}.$$

It is easy to show that P is a normal cone in $C[0,1]$ (with $\gamma=1$). Obviously, for $x,y\in C[0,1]$, $x\preccurlyeq y$ if and only if $x(t)\leqslant y(t)$ for every $t\in[0,1]$. Fix $u_0(t)\equiv 1$ on $[0,1]$. Then

$$P(u_0)=\left\{x\in C[0,1]\colon\ x(t)>\frac{m(t)}{M}\|x\|,\ t\in[0,1]\right\}.$$

For $x\in P$ and $t\in[0,1]$ define the integral operator

$$(Fx)(t)=\int_0^1 k(t,s)f\big(s,x(s)\big)\,ds,$$

where the kernel k is given by (8). Clearly, every fixed point of F is a solution of the problem (1). For $t\in[0,1]$ and $x\in P$ we have in view of (9) and (10)

$$\int_0^1 k(s,s)f\big(s,x(s)\big)\,ds\geqslant\frac{1}{M}\|Fx\|$$

and

$$(Fx)(t)\geqslant m(t)\int_0^1 k(s,s)f\big(s,x(s)\big)\,ds.$$

Thus for every $t\in[0,1]$,

$$(Fx)(t)\geqslant\frac{m(t)}{M}\|Fx\|,$$

which means that $F: P\to P$. Let $\Omega_1=\{x\in C[0,1]\colon\ \|x\|<r_1\}$ and $\Omega_2=\{x\in C[0,1]\colon\ \|x\|<r_2\}$. We can assume that $r_1<r_2$. Clearly F is a completely continuous operator on $P\cap\bar{\Omega}_2$. By (1°) and (9), for $x\in P\cap\partial\Omega_1$ and $t\in[0,1]$, we get

$$(Fx)(t)\leqslant M\int_0^1 k(s,s)\left[M\int_0^1 k(\tau,\tau)\,d\tau\right]^{-1}r_1ds=r_1.$$

Hence $\|Fx\|\leqslant\|x\|$ for $x\in P\cap\partial\Omega_1$. Let $x\in P(u_0)\cap\partial\Omega_2$. Then $\|x\|=r_2$ and $x(t)\in(0,r_2]$ for every $t\in[0,1]$. By (2°), we obtain

$$\begin{aligned}(Fx)(t_0)&\geqslant\int_0^1 k(t_0,s)g(s)h\big(x(s)\big)\,ds=\int_0^1 k(t_0,s)g(s)\frac{h(x(s))}{x^a(s)}x^a(s)\,ds\\&\geqslant\frac{h(r_2)}{r_2^a}\int_0^1 k(t_0,s)g(s)x^a(s)\,ds>\frac{h(r_2)}{r_2^a}\int_0^1 k(t_0,s)g(s)\left[\frac{m(s)\|x\|}{M}\right]^a ds\end{aligned}$$

$$= \frac{h(r_2)}{M^a} \int_0^1 k(t_0, s) g(s) m^a(s)\, ds \geqslant r_2.$$

This gives $\|Fx\| \geqslant \|x\|$ for $x \in P(u_0) \cap \partial\Omega_2$. By Theorem 1, F has a fixed point $x^* \in P$ such that $r_1 \leqslant \|x^*\| \leqslant r_2$. This ends the proof of Theorem 2. $\square$

Example. Consider the problem (1) with

$$f(t,x) = \begin{cases} 16(t+1)x^2(x-3)^2, & (t,x) \in [0,1] \times [0,3], \\ (t+1)(x-3), & (t,x) \in [0,1] \times (3,\infty) \end{cases}$$

and $\alpha = \eta = 1/2$. It is easy to verify that the assumptions of Theorem 2 are satisfied for $r_1 = 1/60$, $r_2 = 1$, $t_0 = 1$, $a = 2$, $g(t) = 16(t+1)$, and $h(x) = x^2(x-3)^2$. Therefore the problem (1) has a positive solution x^* defined on $[0,1]$ such that $1/60 \leqslant \|x^*\| \leqslant 1$.

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